# VECTORS

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**C1A. Vectors & Their Representation :**

Vector quantities are specified by definite magnitude and definite directions. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the **initial point** & B is called the **terminal point.** The magnitude of vector $\overrightarrow{AB}$ is epxressed by $|\overrightarrow{AB}|$.

**C1B. Types of Vectors**

- **Zero or Null Vector :**

A vector of zero magnitude is a zero vector i.e. which has the same initial & terminal point, is called a **Zero Vector.** It is denoted by **O.** The direction of zero vector is indeterminate or arbitrary.

- **Unit Vector :**

A vector of unit magnitude in the direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically, $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

- **Equal Vectors :**

Two vectors are said to be equal if they have the same magnitude, direction & represent the same physical quantity.

- **Collinear Vectors :**

Two vectors are said to be collinear if their directed line segment are parallel irrespective of their directions. Collinear vectors are also called **parallel vectors.** If they have the same direction they are named as **like vectors** otherwise **unlike vectors.**

Symbolically, two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$

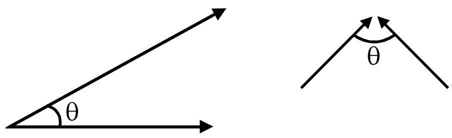
Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

- **Coplanar Vectors :**

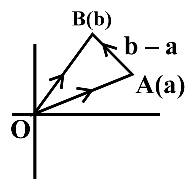
A given number of vectors are called coplanar if their line segments are all parallel to the same plane. Note that **"Two Vectors Are Always Coplanar".**

**C1C. Angle Between two Vectors**

It is the smaller angle formed when the initial points or the terminal points of the two vectors are brought together.
It should be noted that $0^0 \le \theta \le 180^0$.




**C2. Position Vector Of A Point :**




Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$.

If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then

$\overrightarrow{AB} = \vec{b} - \vec{a}$ = position vector of B – position vector of A.

**Practice Problems :**

1. **Find a unit vector in the direction of $\overrightarrow{AB}$, where A(1, 2, 3) and B(4, 5, 6) are the given points.**

2. **Find a vector in the direction of the vector $\vec{a} = (3\hat{i} + \hat{j})$ that has magnitude 5 units.**

3. **If $\vec{a}, \vec{b}$ are the vectors forming consecutive sides of a regular hexagon ABCDEF, express the vectors $\overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EF}, \overrightarrow{FA}, \overrightarrow{AC}, \overrightarrow{AD}, \overrightarrow{AE}$ and $\overrightarrow{CE}$ in terms of $\vec{a}$ and $\vec{b}$.**

**[Answers : (1)** $\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right)$ **(2)** $\left(\frac{15}{\sqrt{10}}\hat{i}+\frac{5}{\sqrt{10}}\hat{j}\right)$ **(3)** $\overrightarrow{CD}=\vec{b}-\vec{a}$ , $\overrightarrow{AF}=\vec{b}-\vec{a}$ ,

$\overrightarrow{AE}=2\vec{b}-\vec{a}$ , $\overrightarrow{CE}=\vec{b}-2\vec{a}$ **]**

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**C3. Direction Ratios and Direction Cosines of a Vector**

Consider a vector $\vec{r}=a\hat{i}+b\hat{j}+c\hat{k}$ . Then, the numbers a, b, c are called the direction ratios of $\vec{r}$ .

Direction cosines of $\vec{r}$ are given by $l=\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\frac{b}{\sqrt{a^2+b^2+c^2}}$ and $n=\frac{c}{\sqrt{a^2+b^2+c^2}}$ .

If *l, m, n* are the direction cosines of a vector then we always have $(l^2+m^2+n^2)=1$.

**Practice Problems :**

1. **If A(1, 2, –3) and B(–1, –2, 1) are two given points in space then find (i) the direction ratios of $\overrightarrow{AB}$ and (ii) the direction cosines of $\overrightarrow{AB}$ .**

**[Answers : (1) (i) (–2, –4, 4) (ii)** $\frac{-1}{3},\frac{-2}{3},\frac{2}{3}$ **]**

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**C4. Distance Formula**

Distance between the two points $A(\vec{a})$ and $B(\vec{b})$ is AB = $|\vec{a}-\vec{b}|$

**Practice Problems :**

1. **Let α, β, γ be distinct real numbers. Prove that the points with position vectors $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ , $\beta\hat{i}+\gamma\hat{j}+\alpha\hat{k}$ , $\gamma\hat{i}+\alpha\hat{j}+\beta\hat{k}$ form an equilateral triangle.**

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**C5. Section Formula**

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A & B then the postion vector of a point which divides AB in the ratio m : n is given by : $\vec{r}=\frac{n\vec{a}+m\vec{b}}{m+n}$.

Note postion vector of mid point of $AB=\frac{\vec{a}+\vec{b}}{2}$

**Practice Problems :**

1. **The position vectors of two vertices and the centroid of a triangle are $\hat{i}+\hat{j}, 2\hat{i}-\hat{j}+\hat{k}$ and $\hat{k}$ respectively. Find the position vector of the third vertex of the triangle ?**

**[Answers : (2)** $-3\hat{i}+2\hat{k}$ **]**

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**C6. Operation on Vectors :**

**C6A. Addition Of Vectors :**

- If two vectors $\vec{a}\ \&\ \vec{b}$ are represented by $\overrightarrow{OA}\ \&\ \overrightarrow{OB}$ , then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$ , where OC is the diagonal of the parallelogram OACB.

- $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ **(commutative)**
- $(\vec{a}+\vec{b})=\vec{a}+(\vec{b}+\vec{c})$ **(associativity)**
- $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$
- $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$
- $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$
- $|\vec{a}-\vec{b}|\ge||\vec{a}|-|\vec{b}||$
- A vector in the direction of the bisector of the angle between the two vectors $\vec{a}\,\&\,\vec{b}$ is $\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is $\lambda(\hat{a}+\hat{b})$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}\,\&\,\vec{b}$ is $\lambda(\hat{a}-\hat{b})$, $\lambda \in R^+$.

**C6B. Multiplication Of A Vector By A Scalar :**

If $\vec{a}$ is a vector & m is a scalar, then $m\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is |m| times that of $\vec{a}$. This multiplication is called Scalar Multiplication. If $\hat{a}$ and $\hat{b}$ are vectors & m, n scalars, then :

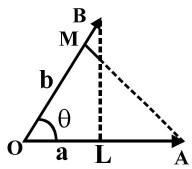
$\mathbf{m(\vec{a})=(\vec{a})m=m\vec{a}}$ $\qquad$ $\mathbf{m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}}$

$\mathbf{(m+n)\vec{a}=m\vec{a}+n\vec{a}}$ $\qquad$ $\mathbf{m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}}$

**C6C. Scalar Product Of Two Vectors :**

Geometrical interpretation of Scalar Product

Let $\vec{a}$ **and** $\vec{b}$ be vector represented by $\overrightarrow{OA}$ **and** $\overrightarrow{OB}$ respectively. Let θ be the angle between $\overrightarrow{OA}$ **and** $\overrightarrow{OB}$.
Draw BL ⊥ OA and AM ⊥ OB.
From ∆s OBL and OAM, we have OL = OB cos θ and OM = OA cos θ. Here OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$ respectively.




Now, $\quad \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{a}|(OB\cos\theta)$

$= |\vec{a}|(OL)$

= (Magnitude of $\vec{a}$) (Projection of $\vec{b}$ on $\vec{a}$) ....(i)

Again $\quad \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{b}|(|\vec{a}|\cos\theta)$

$= |\vec{b}|(OA\cos\theta)$

$= |\vec{b}|(OM)$

= (magnitude of $\vec{b}$) (Projection of $\vec{a}$ and $\vec{b}$ ....(ii)

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

1. i.i = j.j = k.k = 1; i.j = j.k = k.i = 0 ⇒ projection of $\vec{a}$ **on** $\vec{b}=\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

2. if $\vec{a} = a_1 i + a_2 j + a_3 k$ & $\vec{b} = b_1 i + b_2 j + b_3 k$ then $\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

   $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

3. the angle $\phi$ between $\vec{a}$ and $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ $0 \le \phi \le \pi$

4. $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$ $(0 \le \theta \le \pi)$, note that if $\theta$ is acute then $\vec{a}.\vec{b} > 0$ if $\theta$ is obtuse then $\vec{a}.\vec{b} < 0$

5. $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2, \vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative) $\Rightarrow$ $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive)

6. $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $\quad (\vec{a} \ne 0, \vec{b} \ne 0)$

7. $(m\vec{a}).\vec{b} = \vec{a}.(m\vec{b}) = m(\vec{a}.\vec{b})$ (associative) where m is scalar.

**Note :**

(i) Maximum value of $\vec{a}.\vec{b}$ is $|\vec{a}||\vec{b}|$

(ii) Minimum value of $\vec{a}.\vec{b}$ is $-|\vec{a}||\vec{b}|$

(iii) Any vector $\vec{a}$ can be written as, $\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

**<u>Practice Problems :</u>**

**1.** **Find the projection of $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ on $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.**

**2.** **If $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3, |\vec{b}| = 5, |\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.**

**3.** **If $\hat{a}$ and $\hat{b}$ are unit vectors and $\theta$ is the angle between them, prove that $\sin\dfrac{\theta}{2} = \dfrac{1}{2}|\hat{a} - \hat{b}|$.**

**4.** **If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ then find the value of $(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$.**

**5.** **For any vector $\vec{a}$ in space, show that $(\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k} = \vec{a}$.**

**6.** **If $\vec{a} = (2\hat{i} + 2\hat{j} + 3\hat{k}), \vec{b} = (-\hat{i} + 2\hat{j} + \hat{k})$ and $\vec{c} = (3\hat{i} + \hat{j})$ such that $(\vec{a} + \lambda\vec{b}) \perp \vec{c}$, then find the value of $\lambda$.**

**7.** **If A(0, 1, 1), B(3, 1, 5) and C(0, 3, 3) be the vertices of a $\Delta$ ABC, show, using vectors, that $\Delta$ ABC is right angled at C.**

**8.** **Show that the points A(2, –1, 1), B(1, –3, –5) and C(3, –4, –4) are the vertices of a right-angled triangle. Also, find the remaining angles of the triangle.**

**9.** **Let $(\hat{i} + \hat{j} + \hat{k}), (2\hat{i} + 5\hat{j}), (3\hat{i} + 2\hat{j} - 3\hat{k})$ and $(\hat{i} - 6\hat{j} - \hat{k})$ be the position vectors of points A, B, C, D respectively. Find the angle between AB and CD. Hence, show that AB || CD.**

**10.** **Express the vector $\vec{a} = (5\hat{i} - 2\hat{j} + 5\hat{k})$ as sum of two vectors such that one is parallel to the vector $\vec{b} = (3\hat{i} + \hat{k})$ and the other is perpendicular to $\vec{b}$.**

**11.** **The dot products of a vector with the vectors $(\hat{i} + \hat{j} - 3\hat{k}), (\hat{i} + 3\hat{j} - 2\hat{k})$ and $(2\hat{i} + \hat{j} + 4\hat{k})$ are 0, 5, 8 respectively. Find the vector.**

**12.** **Find the projection of $\vec{b} = 2\hat{i} + 3\hat{j} - 2\hat{k}$ in the direction of vector $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$. What is the vector determined by the projection.**

13. **If $\vec{a} = 4\hat{i} + 6\hat{j}$ and $\vec{b} = 3\hat{j} + 4\hat{k}$, find the vector form of the component of $\vec{a}$ and $\vec{b}$.**

14. **Let $\vec{u}, \vec{v}, \vec{w}$ be vectors such that $\vec{u} + \vec{v} + \vec{w} = 0$. If $|\vec{u}| = 3, |\vec{v}| = 4, |\vec{w}| = 5$, then prove that the value of $\vec{u}.\vec{v} + \vec{v}.\vec{u} + \vec{w}.\vec{u}$ is – 25.**

**[Answers : (1) $\frac{5}{\sqrt{6}}$ (2) $60^0$ (4) –3/2 (6) 8 (8) $A = \cos^{-1}\left(\sqrt{\frac{35}{41}}\right)$ and $B = \cos^{-1}\left(\sqrt{\frac{6}{41}}\right)$ (10)**

**$(6\hat{i} + 2\hat{k})$ and $(-\hat{i} - 2\hat{j} + 3\hat{k})$ (11) $\hat{i} + 2\hat{j} + \hat{k}$ (12) $\frac{1}{7}(\hat{i} + 2\hat{j} + 3\hat{k})$ (13) $\overrightarrow{OP} = \frac{18}{5}\hat{b} = \frac{18}{25}(3\hat{j} + 4\hat{k})$ ]**

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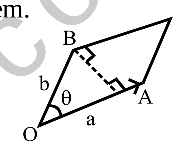
**C6D. Vector Product Of Two Vectors :**

1. If $\vec{a}$ and $\vec{b}$ are two vectors & θ is the angle between them then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\vec{n}$, where $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}$ & $\vec{n}$ forms a right handed screw system.




2. Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$

3. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}; \hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

4. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

5. $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (not commutative)

6. $(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b}) = m(\vec{a} \times \vec{b})$ (associative) where m is a scalar.

7. $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (distributive)

8. $\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a} \times \vec{b}$ are parallel (collinear) $(\vec{a} \neq 0, \vec{b} \neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

9. Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\hat{n} = \pm\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

- A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\frac{r(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$

- If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

- If $\vec{a}, \vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle ABC = $\frac{1}{2}\left[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right]$.

  The points A, B & C are collinear if $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$.

- Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

- Lagrange's Identity : for any two vectors $\vec{a}$ **and** $\vec{b}$; $(\vec{a} \times \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

**Practice Problems :**

1. **Find the area of the parallelogram whose diagonals are represented by the vectors $\vec{d}_1 = (2\hat{i} - \hat{j} + \hat{k})$ and $\vec{d}_2(3\hat{i} + 4\hat{j} - \hat{k})$.**

2. **Find the area of the triangle whose adjacent sides are determined by the vectors $\vec{a} = (-2\hat{i} - 5\hat{k})$ and $\vec{b} = (\hat{i} - 2\hat{j} - \hat{k})$.**

3. **Show that the points whose position vectors are $(5\hat{i} + 6\hat{j} + 7\hat{k})$, $(7\hat{i} - 8\hat{j} + 9\hat{k})$ and $(3\hat{i} + 20\hat{j} + 5\hat{k})$ are collinear.**

4. **Show that the points having position vectors $(\vec{a} - 2\vec{b} + 3\vec{c})$, $(-2\vec{a} + 3\vec{b} + 2\vec{c})$, $(-8\vec{a} + 13\vec{b})$ are collinear, whatever be $\vec{a}, \vec{b}, \vec{c}$.**

5. **Prove that $|\vec{a} \times \vec{b}|^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$.**

6. **Prove that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$.**

7. **If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, prove that $(\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c}) = (\vec{c} \times \vec{a})$.**

8. **Prove that the points A, B, C with position vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear if and only if $(\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) + (\vec{a} \times \vec{b}) = \vec{0}$.**

9. **If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, show that $(\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$, it being given that $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.**

10. **If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a}.\vec{c} = 3$.**

**[Answers : (1) $\frac{1}{2}\sqrt{155}$ sq.units (2) $\frac{1}{2}\sqrt{165}$ sq.units ]**

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**C6E. Scalar Triple Product :**

- The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as : $\vec{a} \times \vec{b}.\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where $\theta$ is the angle between $\vec{a}$ & $\vec{b}$ & $\phi$ is the angle between $\vec{a} \times \vec{b}$ & $\vec{c}$. It is also written as $[\vec{a}\ \vec{b}\ \vec{c}]$ and spelled as box product.
- Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}$, $\vec{b}$ & $\vec{c}$ i.e. $V = [\vec{a}\ \vec{b}\ \vec{c}]$.
- In a scalar triple product the position of dot & cross can be interchanged i.e.

  $\vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$
- $\vec{a}.(\vec{b} \times \vec{c}) = -\vec{a}.(\vec{c} \times \vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$
- If $\vec{a} = a_1 i + a_2 j + a_3 k$; $\vec{b} = b_1 i + b_2 j + b_3 k$ & $\vec{c} = c_1 i + c_2 j + c_3 k$ then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

In general, if $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}; \vec{b} = b_1\vec{l} + b_2\vec{m} - b_3\vec{n}$ & $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\,\vec{m}\,\vec{n}]$; **where** $\vec{l}, \vec{m}$ **&** $\vec{n}$ are non coplanar vectors

- If $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow$ $[\vec{a}\,\vec{b}\,\vec{c}] = 0$.
- Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ .
- If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar then $[\vec{a}\,\vec{b}\,\vec{c}] > 0$ for right handed system & $[\vec{a}\,\vec{b}\,\vec{c}] < 0$ for left handed system.
- $[i\,j\,k] = 1 \Rightarrow K[\vec{a}\,\vec{b}\,\vec{c}] = K[\vec{a}\,\vec{b}\,\vec{c}] = K[\vec{a}\,\vec{b}\,\vec{c}] \quad \Rightarrow [(\vec{a} + \vec{b})\vec{c}\,\vec{d}] = [\vec{a}\,\vec{c}\,\vec{d}] - [\vec{b}\,\vec{c}\,\vec{d}]$
- The volume of the tetrahedron OABC with O as origin & the pv's of A, B and C being $\vec{a}, \vec{b}$ & $\vec{c}$ respectively is given by $V = \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$
- The position vector of the centroid of a tetrahedron in the pv's of its vertices are $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}[\vec{a} + \vec{b} + \vec{c} + \vec{d}]$.

  Note that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

  Remember that : $[\vec{a} - \vec{b}\ \ \vec{b} - \vec{c}\ \ \vec{c} - \vec{a}] = 0$ & $[\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

**Practice Problems :**

**1.** **Find the volume of the parallelopiped whose edges are represented by the vectors**

(a) $\vec{a} = (2,-3,4), \vec{b} = (1,2,-1), \vec{c} = (3,-1,2)$

(b) $\vec{a} = (2,-3,0), \vec{b} = (1,1,-1), \vec{c} = (3,0,-1)$

**2.** **The volume of the parallelopiped whose edges are represented by** $-12\hat{i} + \lambda\hat{k}, 3\hat{j} - \hat{k}, 2\hat{i} + \hat{j} - 15\hat{k}$ **is 540. Find the value of λ.**

**3.** **If the vectors** $a\hat{i} + \hat{j} + \hat{k}, \hat{i} + b\hat{j} + \hat{k}$ **and** $\hat{i} + \hat{j} + c\hat{k}$ **(a ≠ b, c ≠ 1) are coplanar, then prove that the value of** $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ **is 1.**

**4.** **If** $\vec{a}, \vec{b}, \vec{c}$ **be three vectors, prove that**

(i) $[\vec{a} + \vec{b}\quad \vec{b} + \vec{c}\quad \vec{c} + \vec{a}] = 2[\vec{a}\quad \vec{b}\quad \vec{c}]$

(ii) $[\vec{a} - \vec{b}\quad \vec{b} - \vec{c}\quad \vec{c} - \vec{a}] = 0$

(iii) $[\vec{a} \times \vec{b}\ \ \vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$

**5.** **Find the volume of a parallepiped whose sides are given by :** $2\vec{i} - 3\vec{j} + 4\vec{k}, \vec{i} + 2\vec{j} - \vec{k}$ **and** $3\vec{i} - \vec{j} + 2\vec{k}$

**6.** **Find λ so that the following vectors are coplanar** $2\vec{i} - 4\vec{j} + 5\vec{k}, \vec{i} - \lambda\vec{j} + \vec{k}, 3\vec{i} + 2\vec{j} - 5\vec{k}$ **.**

7. **Find the volume of a parallelopiped whose sides are given by** $-3\vec{i}+7\vec{j}+5\vec{k},\ -5\vec{i}+7\vec{j}-3\vec{k}$ **and** $7\vec{i}-5\vec{j}-3\vec{k}$.

8. **Prove that :** $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right]=2\left[\vec{a}\,\vec{b}\,\vec{c}\right]$.

9. **Prove that :** $\left[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}\right]=0$

10. **Find the constant $\lambda$ so that the vector** $\vec{a}=2\hat{i}-\hat{j}+\hat{k},\vec{b}=\hat{i}+2\hat{j}-3\hat{k},\vec{c}=3\hat{i}+\lambda\hat{j}+5\hat{k}$ **are coplanar.**

11. **If** $\vec{x}.\vec{A}=0,\vec{x}.\vec{B}=0,\vec{x}.\vec{C}=0$ **for some non-zero vector** $\vec{x}$ **, then prove that** $\left[\vec{A}\ \vec{B}\ \vec{C}\right]=0$.

**[Answers : (5) 7 (6) 26/25 (7) 264]**

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## C6F. Vector Triple Product :

Let $\vec{a},\vec{b},\vec{c}$ be any three vectors, then the expression $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector & is called a vector triple product.

**Geometrical Interpretation of $\vec{a}\times(\vec{b}\times\vec{c})$**

Consider the expression $\vec{a}\times(\vec{b}\times\vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ & $(\vec{b}\times\vec{c})$. Now $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector perpendicular to the plane containing $\vec{a}$ & $(\vec{b}\times\vec{c})$ but $(\vec{b}\times\vec{c})$ is a vector perpendicular to the plane containing $\vec{b}$ & $\vec{c}$, therefore $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector which lies in the plane of $\vec{b}\times\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a}\times(\vec{b}\times\vec{c})$ in terms of $\vec{b}$ & $\vec{c}$ i.e. $\vec{a}\times(\vec{b}\times\vec{c}) = x\vec{b}+y\vec{c}$ where x & y are scalars.

- $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$
- $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$
- $(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$, in general

**Practice Problems :**

1. **Prove that** $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$.

2. **If** $\vec{a}=2\hat{i}+2\hat{j}-\hat{k},\vec{b}=3\hat{i}-\hat{j}-\hat{k}$ **and** $\vec{c}=\hat{i}+2\hat{j}-3\hat{k}$**, verify that** $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$.

3. **For any non-zero vector** $\vec{a}$**, show that** $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$.

4. **If** $\vec{a},\vec{b},\vec{c}$ **are three vectors, prove that** $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$ **if and only if** $\vec{b}(\vec{c}\times\vec{a})=\vec{0}$.

5. **Given that** $\vec{a}=\hat{i}+2\hat{j}+\hat{k},\hat{b}=3\hat{i}+2\hat{j}-7\hat{k}$ **and** $\vec{c}=5\hat{i}+6\hat{j}-5\hat{k}$**, verify that** $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$.

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## C7. Reciprocal System Of Vectors :

If $\vec{a},\vec{b},\vec{c}$ & $\vec{a}',\vec{b}',\vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}.\vec{a}'=\vec{b}.\vec{b}'=\vec{c}.\vec{c}'=1$ then the two systems are called Reciprocal System of vectors.

Note : $\vec{a}=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{b}=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{c}=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

## C8. Linear Combinations :

Given a finite set of vectors $\vec{a},\vec{b},\vec{c}$ ............. then the vector $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}+.....$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}$ for any x, y, z,........ $\in$ R. We have the following results :

**(a)** If $\vec{a}, \vec{b}$ are non zero, non-collinear vectors then $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x'; y = y'$

**(b)** **Fundamental Theorem :** Let $\vec{a}, \vec{b}$ be non-zero, non collinear vectors. Then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquelly as a linear combinations of $\vec{a}, \vec{b}$.

**(c)** If $\vec{a}, \vec{b}, \vec{c}$ are non-zero, non-coplanar vectors then :

$$x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c} \Rightarrow x = x', y = y', z = z'$$

**(d)** **Fundamental Theorem In space :** Let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquly expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. There exist some unique x, y $\in$ R such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.

**(e)** If $\vec{x}_1, \vec{x}_2 ......\vec{x}_n$ are n non zero vectors, & $k_1$, $k_2$, ........ $k_n$ are n scalars & if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + ......k_n\vec{x}_n = 0 \Rightarrow k_1 = 0, k_2 = 0......k_n = 0$ then we say that vectors $\vec{x}_1, \vec{x}_2 ......\vec{x}_n$ are **LINEARLY INDEPENDENT VECTORS.**

**(f)** If $\vec{x}_1, \vec{x}_2 ......\vec{x}_n$ are not **LINEARLY INDEPENDENT** then they are said to be **LINEARLY INDEPENDENT VECTORS** i.e. if $k_1\vec{x}_1 + k_2\vec{x}_2 + ......k_n\vec{x}_n = 0$ & if there atleast one $k_r \neq 0$ then $\vec{x}_1, \vec{x}_2 ......\vec{x}_n$ are said to be

**Linearly Dependent :**

**Note 1 :** If

$$k_r \neq 0; k_1\vec{x}_1 + k_2\vec{x}_2 + k_3\vec{x}_3 + .... + k_r\vec{x}_r + .... + k_n\vec{x}_n = 0$$

$$-k_r\vec{x}_r = k_1\vec{x}_1 + k_2\vec{x}_2 + .... + k_{r-1}.\vec{x}_{r-1} + k_{r+1} + .... + k_n\vec{x}_n$$

$$-k_r\frac{1}{k_r}\vec{x}_r = k_1\frac{1}{k_r}\vec{x}_1 + k_2\frac{1}{k_r} + .... + k_{r-1}.\frac{1}{k_r}\vec{x}_{r-1} + .... + k_n\frac{1}{k_r}\vec{x}_n$$

$$\vec{x}_r = c_1x_1 + c_2x_2 + ..... + c_{r-1}\vec{x}_{r-1} + c_r\vec{x}_{r-1} + .... + c_n\vec{x}_n$$

i.e. $\vec{x}_r$ is expressed as a linear combination of vectors.

$\vec{x}_1, \vec{x}_2, ..........\vec{x}_{r-1}, \vec{x}_{r+1}, ....\vec{x}_n$

Hence $\vec{x}_r$ **with** $\vec{x}_1, \vec{x}_2, .....\vec{x}_{r-1}, \vec{x}_{r+1} ....\vec{x}_n$ forms a linearly dependent set of vectors.

**Note 2 :**

- If $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$ then $\vec{a}$ is expressed as a **Linear Combination** of vectors $\hat{i}, \hat{j}, \hat{k}$ form a linearly dependent set of vectors. In general, every set of four vectors is a linearly dependent system.
- $\hat{i}, \hat{j}, \hat{k}$ are **Linearly Independent** set of vectors. For

$$K_1\hat{i} + K_2\hat{j} + K_3\hat{k} = 0 \Rightarrow K_1 = K_2 = K_3 = 0$$

- Two vectors $\vec{a}$ & $\vec{b}$ are linearly dependent $\Rightarrow$ $\vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a} \times \vec{b} = 0$ $\Rightarrow$ linear dependence of $\vec{a}$ & $\vec{b}$. Conversely if $\vec{a}$ & $\vec{b}$ then $\vec{a}$ & $\vec{b}$ are linearly independent.

**Note : Test Of Collinearity :**

Three points A, B, C with position vectors $\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that; $\mathbf{x\vec{a}+y\vec{b}+z\vec{c}=0}$, where x + y + z = 0.

**Note : Test Of Coplanarity :**

Four points A, B, C, D with position vectors $\vec{\mathbf{a}},\ \vec{\mathbf{b}},\ \vec{\mathbf{c}},\vec{\mathbf{d}}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $\mathbf{x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d}=0}$ where, x + y + z + w = 0.

**<u>Practice Problems :</u>**

1. **Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ , $\vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors and $|\vec{c}|=\sqrt{3}$ , then find the value of $\alpha$ and $\beta$ ?**